Let $x := \log_a b, \quad y := \log_b c, \quad z := \log_c a$. Then $xyz = \log_a b \log_b c \log_c a = 1$, and the condition $a, b, c \in (0,1)$ or $a, b, c \in (1, \infty)$ implies that $x, \ y, \ z > 0$.

The original inequality may be rephrased as:

$$\frac{x+y}{m+z^{-1}n} + \frac{y+z}{m+x^{-1}n} + \frac{z+x}{m+y^{-1}n} \geq \frac{6}{m+n}, \quad xyz = 1, \quad x, y, z > 0, \tag{1}$$

or as

$$\frac{3}{\sum_{cyc}\left(\frac{m+z^{-1}n}{x+y}\right)^{-1}} \leq \frac{m+n}{2}.$$

Since the harmonic mean is less than or equal to the geometric mean,

$$\frac{3}{\sum_{cyc}\left(\frac{m+z^{-1}n}{x+y}\right)^{-1}} \leq \sqrt[3]{\frac{m+z^{-1}n}{x+y}\frac{m+x^{-1}n}{y+z}\frac{m+y^{-1}n}{z+x}}.$$

Hence it is enough to prove (2):

$$\frac{m+z^{-1}n}{x+y}\frac{m+x^{-1}n}{y+z}\frac{m+y^{-1}n}{z+x} \leq \frac{(m+n)^3}{8},$$

$$\frac{1}{xyz}\frac{(n+mz)(n+mx)(n+my)}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8},$$

$$\frac{(n+mz)(n+mx)(n+my)}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8}. \tag{2}$$

Further simplification of (2) results in

$$\frac{n^3+mn^2x+mn^2y+mn^2z+m^2nxy+m^2nxz+m^2nyz+m^3xyz}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8}$$

$$\frac{n^3+mn^2(x+y+z)+m^2n(xy+yz+xz)+m^3}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8} \tag{3}$$

Equating the left and right sides of (3) shows that the inequality (3) is equivalent to (4) and (5):

$$\frac{x+y+z}{(x+y)(x+z)(y+z)} \leq \frac{3}{8}, \tag{4}$$

$$\frac{xy+yz+xz}{(x+y)(x+z)(y+z)} \leq \frac{3}{8}. \tag{5}$$

We now use the p, q, r notation:

$$p := x+y+z,$$
$$q := xy+yz+zx,$$
$$r := xyz.$$

In this notation, (4) and (5) become

$$\frac{p}{pq-r} \leq \frac{3}{8}, \tag{6}$$

$$\frac{q}{pq-r} \leq \frac{3}{8}. \tag{7}$$

In our case $r=1$, which implies (by AM-GM inequality) that $p \geq 3$ and $q \geq 3$. Now proving (4) and (5) is straightforward:

$$\begin{aligned} \frac{p}{pq-1} &\leq \frac{3}{8}, \\ 3pq-3-8p &\geq 0, \\ 3pq-3-8p &\geq p-3 \geq 0. \end{aligned}$$

$$\begin{aligned} \frac{q}{pq-1} &\leq \frac{3}{8}, \\ 3pq-3-8q &\geq 0, \\ 3pq-3-8q &\geq q-3 \geq 0. \end{aligned}$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Note that since $\log_a b = \dfrac{\ln b}{\ln a}$ and $a,b,c \in (0,1)$ or $a,b,c \in (0,1)$, all the logarithms in the proposed inequality are positive, so the right-hand side is positive.

We will apply the following parametrized Nesbitt's inequality (see reference 1, theorem 7).

Let $x,\ y,\ z,\ tx+ky+lz,\ ty+kz+lx,\ tz+kx+ly$ be positive real numbers and let $-k-l<t<\dfrac{k+l}{2}$.

Then $\dfrac{x}{tx+ky+lz}+\dfrac{y}{ty+kz+lx}+\dfrac{z}{tz+kx+ly} \geq 3t+k+l. \qquad (1)$

We will consider two inequalities, from which the stated problem will follow.

$$\frac{\log_a b}{m+n\log_a c}+\frac{\log_b c}{m+n\log_b a}+\frac{\log_c a}{m+n\log_c b} \geq \frac{3}{m+n} \qquad (2)$$

$$\frac{\log_b c}{m+n\log_a c}+\frac{\log_c a}{m+n\log_b a}+\frac{\log_a b}{m+n\log_c b} \geq \frac{3}{m+n}. \qquad (3)$$

Notice that the right-hand side of (2) is

$$RHS = \frac{\ln b}{m\ln a+n\ln c}+\frac{\ln c}{m\ln b+n\ln a}+\frac{\ln a}{m\ln c+n\ln b} \geq \frac{3}{m+n}$$

by the parametrized Nesbitt's inequality with $t=0$, $k=m$ and $l=n$, and $x=\ln b$, $y=\ln c$, and $z=\ln a$. It also should be noticed that in the last expression we may assume that all the ln's are positive.

Now, the right-hand side of (3) is

$$RHS = \frac{\ln a\ln c}{m\ln a\ln b+n\ln b\ln c}+\frac{\ln a\ln b}{m\ln b\ln c+n\ln a\ln c}+\frac{\ln b\ln c}{m\ln a\ln c+n\ln a\ln b} \geq \frac{3}{m+n}$$

by the parametrized Nesbitt's inequality with $t=0$, $k=m$ and $l=n$, and $x=\ln a\ln c$, $y=\ln a\ln b$, and $z=\ln b\ln c$.

References:

(1) Shanhe Wu and Ovidiu Furdui, *A note on a conjectured Nesbitt type inequality*, Taiwanese Journal of Mathematics, 15 (2) (2011), 449-456.

Solution 3 by Soumitra Mandal, Chandar Nagore, India

$$\sum_{cyc}\frac{\log_a b+\log_b c}{m+n\log_a c}=\sum_{cyc}\frac{\log b+\frac{\log a\cdot\log c}{\log b}}{m\log a+n\log c}$$

$$=\sum_{cyc}\frac{\log b}{m\log a+n\log c}+\sum_{cyc}\frac{\frac{\log a\cdot\log c}{\log b}}{m\log a+n\log c}$$

$$=\sum_{cyc}\frac{(\log b)^2}{n\log a\cdot\log b+n\log c\cdot\log b}+\sum_{cyc}\frac{\left(\frac{1}{\log b}\right)^2}{\frac{m}{\log b\cdot\log c}+\frac{n}{\log b\cdot\log a}}$$

$$\overbrace{\geq}^{BERGSTROM}\frac{(\log a+\log b+\log c)^2}{(m+n)(\log a\cdot\log b+\log b\cdot\log c+\log c\cdot\log a)}+$$

$$+\frac{\left(\frac{1}{\log a}+\frac{1}{\log b}+\frac{1}{\log c}\right)^2}{(m+n)\left(\frac{1}{\log a\cdot\log b}+\frac{1}{\log b\cdot\log c}+\frac{1}{\log c\cdot\log a}\right)}\geq\frac{3}{m+n}+\frac{3}{m+n}=\frac{6}{m+n}$$

Editor's Comments: **Anna V. Tomova of Varna, Bulgaria** approached the solution as follows: She showed that the left hand side of the inequality can be put into the canonical form of $X+Y+\dfrac{1}{XY}$. She then showed that this canonical form has a global minimum at $(1,1)$, forcing it to have a minimal value of 3, and working with this she produced the final result.

Bruno Salgueiro Fanego of Viveiro, Spain noted that the stated problem is a specific case of a more general result. Namely: If $x,y,z\in(0,\infty)$ and $xyz=1$, then

$$\frac{x+y}{m+\dfrac{n}{z}}+\frac{y+z}{m+\dfrac{n}{x}}+\frac{z+x}{m+\dfrac{n}{y}}\geq\frac{6}{m+n}.$$

He proved the more general result, and applied it to the specific case.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray of Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Shravan Sridhar, Udupi, India; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova of Varna, Bulgaria, and the proposer.

5454: *Proposed by Arkady Alt, San Jose, CA*

Prove that for integers k and l, and for any $\alpha,\beta\in\left(0,\frac{\pi}{2}\right)$, the following inequality holds:

$$k^2\tan\alpha+l^2\tan\beta\geq\frac{2kl}{\sin(\alpha+\beta)}-\left(k^2+l^2\right)\cot(\alpha+\beta).$$